

Fidelity and level correlations in the transition from regularity to chaos

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Mean fidelity amplitude and parametric energy-energy correlations are calculated exactly for a regular system, which is subject to a chaotic random perturbation. It turns out that in this particular case under the average both quantities are identical. The result is compared with the susceptibility of chaotic systems against random perturbations. Regular systems are more susceptible against random perturbations than chaotic ones.

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I. INTRODUCTION

A. Peres wrote in 1984 a highly influential paper on the stability of quantum wave functions under random fluctuations of the Hamiltonian [1]. He introduced the overlap between the wave-function propagated by a known Hamiltonian with the same wave function propagated by a slightly perturbed one as a measure for stability. This quantity is nowadays known as fidelity amplitude. Its modulus square is called fidelity or Loschmidt echo[2]. Based on heuristic arguments, numerics and physical intuition, he concluded that a regular fluctuation, i. e. a fluctuations obeying the same superselection rules as the original Hamiltonian should have less effect on the stability of the quantum state than a chaotic one, i. e. a perturbation with no additional symmetries.

Later the same question was addressed amongst others [3, 4] using ensemble theory [5]. The perturbation as well as the original Hamiltonian were chosen from a Gaussian random matrix ensemble (RME). Average fidelity amplitude $\langle f(t) \rangle$ was calculated in second order perturbation theory. Exponentiating the result yielded

$$\ln \langle f(t) \rangle = -2\pi \frac{\Gamma}{D} \left(\frac{\tau}{2} + \frac{\tau^2}{\beta} + C_{\text{corr}}(\tau) \right) + \mathcal{O}(\Gamma^2), \quad (1)$$

where D is the mean level spacing of the original Hamiltonian, $\tau = t/t_H$ is time measured in units of Heisenberg time $t_H = 2\pi\hbar/D$ and Γ is the Breit-Wigner spreading width of an unperturbed eigenstate[6]. The first term is recognised as Fermi's golden rule (FGR). The second term is due to spectral fluctuations, i. e. fluctuations in the Hamiltonian which do not affect the eigenstates (called regular fluctuations by Peres). The last term is a correction term which accounts for the spectral correlations of the unperturbed system.

The perturbative result (1) was confirmed experimentally [7–10]. Later it was completed and extended by exact calculations for different choices for the unperturbed

system and for the perturbation. Thereby unexpected features like fidelity revival at Heisenberg time [11] or fidelity freeze [12, 13] for purely off-diagonal perturbations were revealed.

Fidelity is, at least when averaged over a large number of eigenstates, closely related to the parametric energy correlator, respectively to its Fourier transform, the so-called cross form-factor [14–17]. The latter measure the susceptibility of the *spectrum* against fluctuations of the Hamiltonian rather than the susceptibility of the wave-function. A relation between fidelity and these quantities is rather surprising but highly welcome for experimental purposes, since spectral measurements are much easier to perform than measurements of fidelity, which requires in principle knowledge of the entire wave-function.

The above mentioned results were derived and are valid under the assumption that already the unperturbed system has chaotic dynamics. But the case originally considered by Peres, where a regular system is perturbed by a chaotic admixture, is in quantum information devices more relevant. There the unperturbed dynamics is usually well controlled. It has been object of several theoretical and numerical [5, 18–22] studies in the recent years. Therefore it comes as a surprise that for this case a detailed ensemble theoretical analysis akin to the ones performed in Refs.[11, 14, 15] is lacking.

The present work aims at filling this gap. We consider the same situation as Peres did: A regular system is perturbed by fully chaotic fluctuations. We present exact analytic results for the averaged fidelity amplitude and cross form-factor in the corresponding RME. It turns out that in average both quantities are identical! What is more the exact result is identical with the one obtained in exponentiated second order perturbation theory (1).

The result is compared with the (known) results for originally chaotic systems, which are perturbed by random fluctuations, confirming that a regular system is more susceptible to random fluctuations than chaotic ones [18, 21].

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II. DEFINITIONS AND RESULTS

Fidelity amplitude is defined by ($\hbar = 1$)

$$f(t) = \langle \psi | e^{iHt} e^{-iH_0 t} | \psi \rangle. \quad (2)$$

Here H_0 describes the regular system and the fluctuating Hamiltonian is given by

$$H = H_0 + \lambda D V, \quad (3)$$

where V is a chaotic admixture. We average over the spectrum of H_0 in an interval which is large enough to contain a large number N of (unperturbed) levels but small enough such that the mean level spacing D is constant. The strength of the perturbation is of order D . This means that the dimensionless perturbation strength λ as well as a typical matrix element of V are of order one (see Eq. (7)). The parameter λ is related to the Breit-Wigner spreading width via $\Gamma = 2\pi\lambda^2 D$.

Following the work by Berry and Tabor [23] in a generic regular system, like for example a rectangular billiard, the energy levels are distributed in an interval in the same way as independent random numbers. Assuming ergodicity the average over the energy interval can be replaced by an ensemble average over N independent random numbers or likewise over an ensemble of $N \times N$ matrices with uncorrelated eigenvalues $E_m^{(0)}$, $m = 1, \dots, N$ (Poissonian spectrum).

For definiteness we assume the distribution function $w(E_m^{(0)})$ of each eigenvalue of H_0 to be a Gaussian with zero mean and variance $N/2\pi$. In a region of order N around the origin the eigenvalues are uniformly distributed with mean level spacing $D = w(0) = N^{-1/2}$ up to corrections of order $1/N$. This implies a weak form of translation invariance

$$\int dx w(x) f(x+y) = \int dx w(x) f(x) + \mathcal{O}(N^{-1}) \quad (4)$$

for any $y = \mathcal{O}(N^0)$, which will be used frequently.

We choose an incoherent superposition of all eigenstates in the interval as initial state. Equation (2) becomes

$$f(t) = \frac{1}{N} \text{tr} e^{-iHt} e^{iH_0 t}. \quad (5)$$

We define the cross form-factor as

$$\tilde{K}(t) = \frac{1}{N} \text{tr} e^{-iHt} \text{tr} e^{iH_0 t}. \quad (6)$$

It is a purely spectral quantity and contains no information about the wave-function. This definition differs from to one of [14] by a singular contribution at $t = 0$ and for $N \rightarrow \infty$.

The chaotic perturbation V is chosen from a Gaussian random matrix ensemble, defined by its second moments

$$\langle V_{ij} V_{kl} \rangle_V = \delta_{il} \delta_{jk} + \left(\frac{2}{\beta} - 1 \right) \delta_{ik} \delta_{jl}. \quad (7)$$

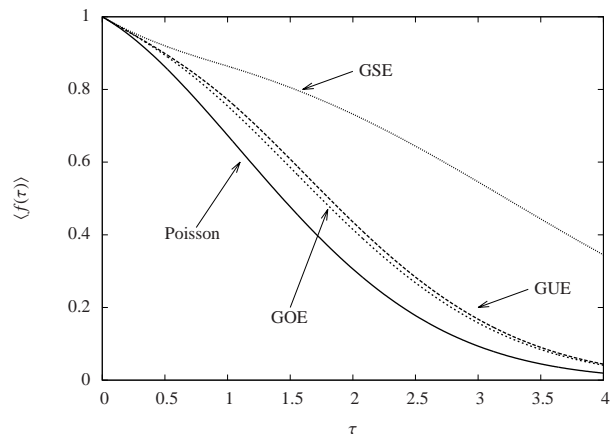


FIG. 1: Plot of equation (9) for $\lambda = 0.1$ (full line) and for a GUE perturbation. The curve is compared with fidelity amplitude of a chaotic original systems H_0 , which is perturbed by a GUE with coupling strength $\lambda = 0.1$. Here H_0 is taken from the GUE (long dashed line, formula from [11]), from the GOE (short dashed line, formula from [15]) and from the GSE (dotted line, formula from [15]).

The Dyson index β labels the three classical ensembles [24]. The Gaussian unitary ensemble (GUE, $\beta = 2$, V Hermitean) is chosen when the perturbation apart from being chaotic breaks time reversal invariance (TRI). The Gaussian orthogonal ensemble (GOE, $\beta = 1$, V real symmetric) respectively the Gaussian symplectic ensemble (GSE, $\beta = 4$, V Hermitean selfdual) are chosen if the perturbation is chaotic but time reversal invariant. The GOE applies for integer spin and the GSE for half-integer spin. The variance of a typical matrix element of V is of order one. This means that the weak translation invariance (4) applies and the density of states of H_0 remains unaffected by the perturbation. In the following angular brackets denote the above defined averages over both V and H_0 .

We now state our main results. First, in the limit $N \rightarrow \infty$ and for $t > 0$ average fidelity amplitude and average cross form-factor are identical

$$\langle f(t) \rangle = \langle \tilde{K}(t) \rangle. \quad (8)$$

Second, for $t > 0$ the exact result for the average fidelity amplitude is for all three ensembles

$$\langle f(t) \rangle = e^{-4\lambda^2 \pi^2 \left(\frac{\tau^2}{\beta} + \frac{\tau}{2} \right)}, \quad \tau = \frac{t}{t_H}. \quad (9)$$

This result coincides with exponentiated second order perturbation theory (1). For $t = 0$, $\langle \tilde{K}(t) \rangle$ differs from $\langle f(t) \rangle$ by a $\delta(t)$ -contribution.

In Fig. 1 equation (9) is plotted for a GUE perturbation ($\beta = 2$). The curve is compared with the corresponding ones for systems which are originally chaotic and which are perturbed by a GUE. We see that a regular system is less resilient against random perturbations than chaotic

ones. This is in accordance with former results [18, 21] in the literature. Among the chaotic systems the GSE is the most resilient against random fluctuations and the GOE the most susceptible. This indicates a monotonous dependence of fidelity decay on the spectral rigidity of the original system, as quantified by the parameter β .

III. CALCULATION OF FIDELITY AMPLITUDE AND CROSS FORM-FACTOR

Despite their simplicity, the derivation of the main identities (8) and (9) requires a full fledged supersymmetric calculation. We sketch the main steps.

A. Map onto a supersymmetric matrix model

After Fourier transforming (5) and (6) fidelity amplitude and cross form-factor are expressed in terms of the resolvents $G(E^\pm) = (H - E \pm i\varepsilon)^{-1}$ and $G_0(E^\pm) = (H_0 - E \pm i\varepsilon)^{-1}$ as

$$f(t) = \frac{1}{N} \int \frac{dE_1 dE_2}{(2\pi)^2} e^{i(E_1 - E_2)t} \text{tr} G_0(E_1^-) G(E_2^+) \quad (10)$$

$$\tilde{K}(t) = \frac{1}{N} \int \frac{dE_1 dE_2}{(2\pi)^2} e^{i(E_1 - E_2)t} \text{tr} G_0(E_1^-) \text{tr} G(E_2^+).$$

Here and in the following we assume $t > 0$ explicitly. We use the following fundamental property of the resolvent

$$G_{ij}(E) = \frac{1}{2} \frac{d}{dK_{ij}} \det \left(\frac{H - E + K}{H - E + K} \right) \Big|_{K=0}, \quad (11)$$

where K is a matrix containing source terms. Now both quantities can be expressed as derivatives with respect to source matrices $K^{(1)}$ and $K^{(2)}$ of one and the same generating function

$$Z(t, K^{(1)}, K^{(2)}) = \frac{1}{N} \int \frac{dE_1 dE_2}{(2\pi)^2} e^{i(E_1 - E_2)t} \frac{\det(H_0 - E_1 + K^{(1)}) \det(H - E_2 + K^{(2)})}{\det(H_0 - E_1^- - K^{(1)}) \det(H - E_2^+ - K^{(2)})} \quad (12)$$

as

$$f(t) = \sum_{n,m} \frac{\partial^2}{\partial K_{nm}^{(1)} \partial K_{mn}^{(2)}} Z \Big|_{\substack{K^{(1)}=0 \\ K^{(2)}=0}}$$

$$\tilde{K}(t) = \sum_{n,m} \frac{\partial^2}{\partial K_{nn}^{(1)} \partial K_{mm}^{(2)}} Z \Big|_{\substack{K^{(1)}=0 \\ K^{(2)}=0}}. \quad (13)$$

After writing the determinants as Gaussian integrals over vectors with commuting (denominator) and anticommuting (numerator) entries the average over the perturbation $\langle Z \rangle_V$ can be performed easily. The following standard steps are explained in detail for instance in [25–27]. First a supermatrix is introduced via a Hubbard–Stratonovich transformation and then the differential operators in Eq. (13) are expressed as differential operators

$\Delta_f(\mathbf{J})$ and $\Delta_{\tilde{K}}(\mathbf{J})$ with respect to the entries of a supermatrix \mathbf{J} . Finally the integrals over the vectors can be performed.

After these steps both average fidelity amplitude and average cross form-factor can be written as $\langle f(t) \rangle = \Delta_f(\mathbf{J}) \mathbf{Z}(\mathbf{J}, t)|_{\mathbf{J}=0}$ and $\langle \tilde{K}(t) \rangle = \Delta_{\tilde{K}}(\mathbf{J}) \mathbf{Z}(\mathbf{J}, t)|_{\mathbf{J}=0}$, where the generating function $\mathbf{Z}(\mathbf{J}, t)$ has a supermatrix \mathbf{J} as argument. It is given by the following supersymmetric matrix integral

$$\mathbf{Z}(\mathbf{J}, t) = \frac{1}{N} \int \frac{dE_1 dE_2}{(2\pi)^2} e^{i(E_1 - E_2)t} \tilde{\mathbf{Z}}(\mathbf{J}, E_1, E_2), \quad (14)$$

and

$$\tilde{\mathbf{Z}}(\mathbf{J}, E_1, E_2) = \int d[\sigma] \exp \left(-\frac{\kappa}{2D^2\lambda^2} \text{Str}(\sigma - E_1)^2 \right) \langle \text{Sdet}^{-\kappa} (H_0 \otimes 1_{4\rho} + 1_N \otimes \Sigma(\mathbf{J})) \rangle_{H_0}. \quad (15)$$

The square bracket $\langle \dots \rangle_{H_0}$ denotes an average over the unperturbed Hamiltonian H_0 . We have $(\kappa, \rho) = (1/2, 2)$ for $V \in \text{GOE}$, $(\kappa, \rho) = (1, 1)$ for $V \in \text{GUE}$ and $(\kappa, \rho) = (1, 2)$ for $V \in \text{GSE}$. Here $\Sigma(\mathbf{J})$ is a $4\rho \times 4\rho$ supermatrix

$$\Sigma(\mathbf{J}) = \begin{bmatrix} \sigma^- & 0 \\ 0 & E_2^+ 1_{2\rho} \end{bmatrix} - \mathbf{J}. \quad (16)$$

We use the standard definitions of a supertrace $\text{Str} \sigma = \text{tr} A_1 - \text{tr} A_2$ and of a superdeterminant $\text{Sdet} \sigma = \det A_2^{-1} \det(A_1 - \Lambda_1 A_2^{-1} \Lambda_2)$ of a block supermatrix $\sigma = \begin{bmatrix} A_1 & \Lambda_1 \\ \Lambda_2 & A_2 \end{bmatrix}$, where the entries of Λ_1 and Λ_2 are anticommuting. The $2\rho \times 2\rho$ supermatrix σ has the form

$$\begin{pmatrix} a_1 & \lambda_1^* \\ \lambda_1 & ia_2 \end{pmatrix}, \quad \text{GUE}$$

$$\begin{pmatrix} a_1 & a_2 & \lambda_1^* & -\lambda_1 \\ a_2 & a_3 & \lambda_2^* & -\lambda_2 \\ \lambda_1 & \lambda_2 & ia_4 & 0 \\ \lambda_1^* & \lambda_2^* & 0 & ia_4 \end{pmatrix}, \quad \text{GOE}$$

$$\begin{pmatrix} a_1 & 0 & \lambda_1^* & -\lambda_1 \\ 0 & a_1 & \lambda_2^* & -\lambda_2 \\ \lambda_1 & \lambda_2 & ia_2 & ia_3 \\ \lambda_1^* & \lambda_2^* & ia_3 & ia_4 \end{pmatrix}, \quad \text{GSE}. \quad (17)$$

The integration variables a_n , $n = 1 \dots 4$ are real commuting variables and λ_n , $n = 1 \dots 2$ are complex anticommuting variables. The matrix integral extends over all independent entries of σ . The $4\rho \times 4\rho$ supermatrix

$$\mathbf{J} = \begin{bmatrix} J^{11} & J^{12} \\ -J^{21} & -J^{22} \end{bmatrix}, \quad (18)$$

contains the source terms, where each of the $2\rho \times 2\rho$ matrices J^{ij} has the structure of the prototypes (17). In a tedious but straightforward calculation the operators

$\Delta_f(\mathbf{J})$ and $\Delta_{\tilde{K}}(\mathbf{J})$ can be worked out. They are given by

$$\begin{aligned}\Delta_f(\mathbf{J}) &= \frac{1}{(2\rho)^2} \sum_n^{2\rho} \left(\sum_m^\rho \frac{\partial^2}{\partial J_{nm}^{21} \partial J_{mn}^{12}} - \sum_{n=\rho+1}^{2\rho} \frac{\partial^2}{\partial J_{nm}^{21} \partial J_{mn}^{12}} \right) \\ \Delta_{\tilde{K}}(\mathbf{J}) &= \frac{1}{(2\rho)^2} \left(\sum_n^\rho \frac{\partial}{\partial J_{nn}^{11}} - \sum_{n=\rho+1}^{2\rho} \frac{\partial}{\partial J_{nn}^{11}} \right) \\ &\quad \times \left(\sum_n^\rho \frac{\partial}{\partial J_{nn}^{22}} - \sum_{n=\rho+1}^{2\rho} \frac{\partial}{\partial J_{nn}^{22}} \right). \quad (19)\end{aligned}$$

In the following we evaluate the matrix integral (15) in the limit $N \rightarrow \infty$, $(E_1 - E_2)/D = r$, and r finite.

B. Evaluation of Eq. (15)

The matrix integral over σ is expressed in angle-eigenvalue coordinates $\sigma \rightarrow U^{-1}sU$, $d[\sigma] \rightarrow B(s)d[s]d\mu(U)$, where U denotes the supergroup, which diagonalizes σ . The diagonal matrix

$$s = \begin{cases} \text{diag}(s_{B1}, is_{F1}), & \text{GUE,} \\ \text{diag}(s_{B1}, s_{B2}, is_{F1}, is_{F2}), & \text{GOE,} \\ \text{diag}(s_{B1}, s_{B2}, is_{F1}, is_{F2}), & \text{GSE,} \end{cases} \quad (20)$$

contains the Bosonic (s_{Bn}) and Fermionic (is_{Fm}) eigenvalues. The Berezinian $B(s)$ is given by [25, 27]

$$B(s) = \begin{cases} \frac{1}{(s_{B1} - is_{F1})^2}, & \text{GUE} \\ \frac{|s_{B1} - s_{B2}| \delta(s_{F1} - s_{F2})}{(s_{B1} - is_{F1})^2 (s_{B2} - is_{F1})^2}, & \text{GOE} \\ \frac{|s_{F1} - s_{F2}| \delta(s_{B1} - s_{B2})}{(is_{F1} - s_{B1})^2 (is_{F2} - s_{B1})^2}, & \text{GSE.} \end{cases} \quad (21)$$

We proceed by observing that the average over H_0 in the last line of equation (15) only depends on the 4ρ eigenvalues of $\Sigma(\mathbf{J})$. These are only infinitesimally different from the eigenvalues of $\Sigma(0)$. The infinitesimal corrections are obtained by a perturbative expansion. The eigenvalues of $\Sigma(0)$ are the eigenvalues of σ and E_2 which is 2ρ -fold degenerate. We lift this degeneracy by replacing $E_2 \mathbb{1}_{2\rho}$ with the auxiliary matrix $\mathbf{E}^a = \text{diag}(E_{B1}^a, \dots, E_{B\rho}^a, E_{F1}^a, \dots, E_{F\rho}^a)$, such that $\Sigma(0)$ is completely non-degenerate. Now the action of the operators $\Delta_f(\mathbf{J})$ and $\Delta_{\tilde{K}}(\mathbf{J})$ can be mapped in a lengthy but straightforward calculation onto the action of first order differential operators in the eigenvalues of s and

\mathbf{E}^a . We denote them by $G^{f/\tilde{K}}$. They read

$$\begin{aligned}G^f &= \frac{1}{(2\rho)^2} \sum_{n,m}^\rho \frac{\mathcal{D}(s_{Bn}, E_{Bm}^a) + \mathcal{D}(s_{Bn}, -E_{Fm}^a)}{s_{Bn} - E_2} \\ &\quad + \frac{1}{(2\rho)^2} \sum_{n,m}^\rho \frac{\mathcal{D}(is_{Fn}, E_{Fm}^a) + \mathcal{D}(is_{Fn}, -E_{Bm}^a)}{s_{Bn} - E_2} \\ G^{\tilde{K}} &= \frac{1}{(2\rho)^2} \sum_{n,m=1}^\rho \mathcal{D}(s_{Bn}, -is_{Fn}) \mathcal{D}(E_{Bm}^a, E_{Fm}^a), \quad (22)\end{aligned}$$

where we used the abbreviation $\mathcal{D}(x, y) = \partial/\partial x - \partial/\partial y$. The action of the differential operator with respect to the source matrix \mathbf{J} on $\tilde{\mathbf{Z}}(\mathbf{J}, E_1, E_2)$ can be replaced by the action of $G^{f/\tilde{K}}$ as follows

$$\begin{aligned}\Delta_{f/\tilde{K}}(\mathbf{J}) \tilde{\mathbf{Z}}(\mathbf{J}, E_1, E_2) \Big|_{\mathbf{J}=0} &= \\ \int B(s) d[s] e^{-\frac{\kappa}{2D\lambda^2} \text{Str}(s-E_1)^2} G^{f/\tilde{K}} Z_0(s, \mathbf{E}^a) \Big|_{\mathbf{E}^a=E_2 \mathbb{1}_{2\rho}}. \quad (23)\end{aligned}$$

Here $Z_0(s, \mathbf{E}^a)$ is the superdeterminant in the second line of equation (15). The matrices s and \mathbf{E}^a are diagonal, thus the superdeterminant can be written as a ratio of ordinary determinants

$$Z_0(s, \mathbf{E}^a) = \left\langle \left(\frac{\prod_{n=1}^\rho \det(H_0 - is_{Fn}) \det(H_0 - E_{Fn}^a)}{\prod_{n=1}^\rho \det(H_0 - s_{Bn}^-) \det(H_0 - E_{Bn}^{a+})} \right)^\kappa \right\rangle. \quad (24)$$

Since the eigenvalues of H_0 are uncorrelated the ensemble average over H_0 of Z_0 is the N -th power of a single integral

$$Z_0(s, \mathbf{E}^a) = \left(\int dx w(x) R(x, s, \mathbf{E}^a) \right)^N, \quad (25)$$

where R is given by

$$R(x, s, \mathbf{E}^a) = \left(\frac{\prod_{n=1}^\rho (x - is_{Fn})(x - E_{Fn}^a)}{\prod_{n=1}^\rho (x - s_{Bn}^-)(x - E_{Bn}^{a+})} \right)^\kappa. \quad (26)$$

Next we have to calculate the action of $G^{f/\tilde{K}}$ on Z_0 . We observe that at $\mathbf{E}^a = E_2 \mathbb{1}_{2\rho}$ the E_2 dependence of Z_0 drops out and Z_0 can be evaluated in the large N -limit

$$\lim_{N \rightarrow \infty} Z_0(s, \mathbf{E}^a) \Big|_{\mathbf{E}^a=E_2 \mathbb{1}_{2\rho}} = \exp(-i\pi\kappa D N \text{Str} s). \quad (27)$$

We need to calculate the action of $G^{f/\tilde{K}}$ on R . One finds in a straightforward calculation

$$\begin{aligned}G^{f/\tilde{K}} \int dx w(x) R(x, s, \mathbf{E}^a) \Big|_{\mathbf{E}^a=E_2 \mathbb{1}_{2\rho}} &= \\ \sum_{n=1}^\rho \mathcal{D}(s_{Bn}, -is_{Fn}) \int dx w(x) \frac{\tilde{R}(s, x)}{x - E_2^+}, \quad (28)\end{aligned}$$

where

$$\tilde{R}(s, x) = \begin{cases} \sqrt{\frac{(x - is_{F1})(x - is_{F2})}{(x - s_{B1}^-)(x - s_{B2}^-)}} & \text{GOE} \\ \frac{x - is_{F1}}{x - s_{B1}^-} & \text{GUE} \\ \frac{(x - is_{F1})(x - is_{F2})}{(x - s_{B1}^-)(x - s_{B2}^-)} & \text{GSE.} \end{cases} \quad (29)$$

Equation (28) holds for both G^f and for $G^{\tilde{K}}$. From this follows immediately $\langle f(t) \rangle = \langle \tilde{K}(t) \rangle$, which is our first main result (8).

Collecting the former results and expressing everything in terms of the dimensionless energy difference r , we find

$$\langle f(t) \rangle = \int \frac{dr}{(2\pi)^2} e^{2\pi i r \tau} \int d[s] B(s) e^{-\frac{\kappa}{2\lambda^2} \text{Str} s^2 - i\pi \kappa \text{Str} s} \sum_{n=1}^{\rho} \mathcal{D}(s_{Bn}, -is_{Fn}) \int dx w(x) \frac{\tilde{R}(s, x)}{x + r^-}. \quad (30)$$

We recall $\tau = t/t_H$. Absolute convergence of the x -integration and of the s integration is guaranteed by the Gaussian weight functions. We are therefore allowed to interchange order of integration and perform the r integration by the residue theorem. Moreover, we perform an integration by parts of the operator $\sum_{n=1}^{\rho} \mathcal{D}(s_{Bn}, -is_{Fn})$ using that this operator annihilates $\text{Str} s$ and $B(s)$. The result is

$$\langle f(t) \rangle = \frac{i\kappa}{\pi\lambda^2} \int d[s] B(s) e^{-\frac{\kappa}{2\lambda^2} \text{Str} s^2 - i\pi \kappa \text{Str} s} \times \text{Str} s \int dx w(x) e^{-2\pi i x \tau} \tilde{R}(s, x). \quad (31)$$

The remaining integral is simple for the GUE but somewhat tricky for the GOE and for the GSE.

In the GUE-case the x -integration can be performed by the residue theorem and employing weak translation invariance. As a result the remaining integrals over s_{B1} and over s_{F1} decouple. Both are Gaussian integrals. The final result is

$$\langle f(t) \rangle_{\text{GUE}} = e^{-2\lambda^2 \pi^2 (\tau^2 + \tau)}. \quad (32)$$

For the GOE the x -integration is more complicated due to the square roots in \tilde{R} and the more complicated Berezinian $B(s)$. The expression on the r.h.s. of (31) is a fourfold integral over the three eigenvalues of σ and over x . In order to simplify this integral, we use identity (27) of Ref. [28] and thereby extract a GUE contribution from Eq. (31). Thus the average fidelity amplitude is given by the GUE result plus an extra term

$$\langle f(t) \rangle_{\text{GOE}} = \langle f(t) \rangle_{\text{GUE}} + \langle f(t) \rangle_{\text{add}}. \quad (33)$$

After using identity (28) of Ref. [28] and an integration

by parts the extra term reads

$$\begin{aligned} \langle f(t) \rangle_{\text{add}} &= -\frac{i}{2\pi\lambda^2} \int \frac{ds_{B1} ds_{B2} ds_{F1} ds_{F2} |s_{B1} - s_{B2}|}{(s_{B1} - is_{F1})(s_{B2} - is_{F1})} \\ &\times \delta(s_{F1} - s_{F2}) \exp\left(-\frac{1}{4\lambda^2} \text{Str} s^2 - \frac{i\pi}{2} \text{Str} s\right) \\ &\times \left(\sum_{n=1}^{\rho} \mathcal{D}(s_{Bn}, -is_{Fn}) - \frac{\text{Str} s}{s_{B1} - s_{B2}} \mathcal{D}(s_{B1}, s_{B2}) \right) \\ &\int dx w(x) e^{-2\pi i x \tau} \tilde{R}(s, x). \end{aligned} \quad (34)$$

The action of the differential operator in the third line of equation (34) on $\tilde{R}(s, x)$ yields the crucial simplification of the integral

$$\langle f(t) \rangle_{\text{add}} = \frac{i}{2\pi\lambda^2} \int ds_{F1} ds_{B1} ds_{B2} |s_{B1} - s_{B2}| \int dx w(x) \frac{e^{-\frac{1}{4\lambda^2} \text{Str} s^2 - \frac{i\pi}{2} \text{Str} s - 2\pi i x \tau}}{\sqrt{x - s_{B1}^-}^3 \sqrt{x - s_{B2}^-}^3}. \quad (35)$$

Now the (trivial) s_{F1} integration decouples from the rest. Introducing coordinates $v = s_{B1} - s_{B2}$ and $u = (s_{B1} + s_{B2})/2$ and using the weak translation invariance of $w(x)$ it is seen that the x -integration does not depend on u . Thus the u -integration decouples as well. Moreover the integrals over s_{F1} and over u together yield again the GUE result. Thus we can write

$$\begin{aligned} \langle f(t) \rangle_{\text{add}} &= \frac{i}{4\pi D} \langle f(t) \rangle_{\text{GUE}} \int dv |v| \\ &\int dx w(x) \frac{e^{-\frac{v^2}{2\lambda^2}} e^{-2\pi i x \tau}}{\sqrt{x - v^-}^3 \sqrt{x + v^+}^3}. \end{aligned} \quad (36)$$

The x -integration can be performed employing weak translation invariance of $w(x)$

$$\int dx w(x) \frac{e^{-2\pi i x \tau}}{\sqrt{x - v^-}^3 \sqrt{x + v^+}^3} = -2\pi i D \frac{d}{dv} J_0(2\pi \tau v). \quad (37)$$

Here $J_0(x)$ is the zeroth order Bessel function. After yet another integration by parts

$$\begin{aligned} \langle f(t) \rangle_{\text{add}} &= \langle f(t) \rangle_{\text{GUE}} \\ &\times \left(-J_0(0) + \int dv |v| e^{-\frac{v^2}{2\lambda^2}} J_0(2\pi \tau v) \right). \end{aligned} \quad (38)$$

The remaining integral over v is a standard Gradsteyn integral [29]. With $J_0(0) = 1$ we find

$$\langle f(t) \rangle_{\text{add}} = -\langle f(t) \rangle_{\text{GUE}} \left(1 - e^{-2\pi^2 \tau^2 \lambda^2} \right) \quad (39)$$

and finally $\langle f(t) \rangle_{\text{GOE}} = e^{-4\lambda^2 \pi^2 (\tau^2 + \frac{\tau}{2})}$. Again the result coincides with exponentiated second order perturbation theory. Here it is even more surprising than for the GUE since the deceptively simple and compact result is

– at least in the way it was calculated here – the outcome of a complicated conspiracy of terms.

The GSE case is treated like the GOE case. The calculation is simpler since no square root appears in $\tilde{R}(s, x)$. The result is $\langle f(t) \rangle_{\text{GSE}} = e^{-4\lambda^2\pi^2(\frac{\tau^2}{4} + \frac{\tau}{2})}$. Thus we can write the mean fidelity amplitude in all three cases concisely as in equation (9).

IV. CONCLUSION

Using supersymmetry we calculated fidelity amplitude and cross form-factor in a random matrix model for a regular system with a chaotic perturbation. Surprisingly both quantities are identical under ensemble average. The result is a simple exponential of two terms. One term decays quadratically in time and one term decays linearly. Exponentiated second order perturbation theory is exact, indicating that a more direct proof of our results is likely to exist.

The fact that both quantities are identical can be reconciled with the general differential identity [15, 17]

$$\langle f(t) \rangle = -\frac{\beta}{4\pi^2\tau^2} \frac{\partial}{\partial(\lambda_{\parallel}^2)} \langle \tilde{K}(t) \rangle, \quad (40)$$

where only the subspace of the perturbation which is parallel to the original Hamiltonian enters. To understand this we split the matrix space of perturbations $\mathcal{V} \ni V$ into a subspace \mathcal{V}_{\parallel} parallel and a subspace \mathcal{V}_{\perp} perpendicular to the matrix space of the unperturbed Hamiltonian

$\mathcal{H}_0 \ni H_0$ via $V \in \mathcal{V}_{\perp} \Leftrightarrow \text{tr} V H_0 = 0, \forall H_0 \in \mathcal{H}_0$ and $\mathcal{V}_{\parallel} = \mathcal{V} - \mathcal{V}_{\perp}$. Likewise any perturbation can be written as $V = V_{\parallel} + V_{\perp}$, where the parallel part shares the symmetries of H_0 . The model (3) might be generalized to $H = H_0 + D\lambda_{\parallel}V_{\parallel} + D\lambda_{\perp}V_{\perp}$ where coupling strengths λ_{\parallel} and λ_{\perp} might be different.

Since a chaotic perturbation breaks any symmetry of the regular system in our case V_{\parallel} consists in the truncation of V to its diagonal part in the eigenbasis of H_0 . It is intuitively clear and has been shown perturbatively [20] that the diagonal part of the perturbation in the eigenbasis of H_0 is responsible for the Gaussian decay. On the other hand the linear term (FGR) is due to the off-diagonal terms. This suggests to write our main results (8) and (9) in the form

$$\langle \tilde{K}(t) \rangle = \langle f(t) \rangle = e^{-4\pi^2 \left(\frac{\lambda_{\parallel}^2 \tau^2}{\beta} + \frac{\tau \lambda_{\perp}^2}{2} \right)}, \quad (41)$$

which obeys the differential relation (40). Although we have proven equation (41) only for $\lambda_{\parallel} = \lambda_{\perp}$, we conjecture it to hold exactly for arbitrary coupling strength λ_{\parallel} and λ_{\perp} . This conjecture is backed by a perturbative calculation akin to that leading to Eq. (1). A rigorous proof remains a challenge for the future.

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